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Perturbative ultraviolet and infrared dynamics of noncommutative quantum field theory ^a

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Abstract

Perturbative aspects of ultraviolet and infrared dynamics of noncommutative quantum field theory is examined in detail. It is observed that high loop momentum contribution to the nonplanar diagram develops a new infrared singularity with respect to the external momentum. This singular behavior is closely related to that of ultraviolet divergence of planar diagram. It is also shown that such a relation is precise in noncommutative Yang-Mills theory, but the same feature does not persist in noncommutative generalization of QED.

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1 Introduction

The primary purpose here is to observe the perturbative aspects of noncommutative quantum field theory. After viewing in Sec. 2 one motivation why noncommutative quantum field theory becomes interesting, we argue ultraviolet (UV) property derived from perturbative consideration [1, 2] with an introduction to perturbative framework of noncommutative quantum field theory in Sec. 3. In Sec. 4, we examine the infrared (IR) aspects, and show how it is closely related to UV ones, especially in noncommutative Yang-Mills (NCYM) theory. Final section is devoted to the discussion and conclusion.

2 Motivation

Noncommutative field theory appears in the matrix models [3, 4]. The matrix model conjecture [5, 6, 7] is intended to provide a constructive definition of the superstring theory and to extract nonperturbative consequences of the interacting superstring dynamics, which will enable us to ask whether string theory is real or not. However, for instance, IIB matrix model [6] does not have any dimensionless coupling constant which can be decreased at will. Therefore, the direct perturbative analysis is not available in that model.

One way to see appearance of noncommutative Yang-Mills theory from matrix model is to expand IIB matrix model action around BPS solution [4]

$$[X^\mu, X^\nu] = -iC^{\mu\nu}1_N. \tag{1}$$

where the size of bosonic matrices X^M ($M = 1, \dots, 10$) is taken to be infinite. $C^{\mu\nu} = -C^{\nu\mu}$ denotes the abelian part of the field strength $F_{\mu\nu}$, where μ is restricted to $1, \dots, 4$. Reminding that IIB matrix proposes that the eigenvalues of X^M constitutes the points of the universe, at least semiclassically, the above relation (1) implies that the location of each point x^μ is uncertain in those four directions:

$$|x^\mu| |x^\nu| \geq 2\pi |C^{\mu\nu}| \quad \text{for } \mu \neq \nu, \tag{2}$$

and that $C^{\mu\nu}$ characterizes the minimal area of accuracy in each two-dimensional plane.

IIB matrix model action gives an action with respect to the fluctuation a_μ (and the other six bosonic coordinates and fermionic variables) around the previous BPS solution (1), where $X^\mu = X_{(0)}^\mu + C^{\mu\nu} a_\nu$ and $X_{(0)}^\mu$ is the classical part satisfying eq. (1). Indeed, through the map called as “Weyl correspondence”, the system can be described in terms of a four-dimensional field theory (See Ref. [4] on its detail.). The resulting theory is $\mathcal{N} = 4$ supersymmetric noncommutative Yang-Mills (NCYM) theory, where the fields in the action are multiplied by the star-product defined by

$$(f * g)(x) = \exp\left(\frac{1}{2i}\partial_\mu C^{\mu\nu}\partial'_\nu\right) f(x) g(x') \Big|_{x' \rightarrow x}, \quad (3)$$

where $C^{\mu\nu}$ is the parameter appearing before. For the coordinates, for instance, we obtain

$$x^\mu * x^\nu - x^\nu * x^\mu = -iC^{\mu\nu}. \quad (4)$$

This algebraic relation is isomorphic to the original algebra (1) satisfied by the background represented by matrices.

Now the coupling constant g_{NCYM} in the resulting Yang-Mills theory is given by

$$g_{\text{NCYM}}^2 = \frac{4\pi^2 g_{\text{IIB}}^2}{C^2}. \quad (5)$$

Here g_{IIB} is the coupling constant of the original matrix model and also has dimension of the length squared. In the canonical basis of $C^{\mu\nu}$, it has been assumed that $C^{\mu\nu} = C (i\sigma_2 \otimes 1 + 1 \otimes i\sigma_2)$ for simplicity. Thus, by taking $C^{\mu\nu}$ sufficiently large compared to g_{IIB} , we get weak coupling NCYM theory. Hence, we can investigate the dynamics of matrix model by analyzing the quantum mechanical aspect of NCYM theory. The challenge is to show the existence of gravity and string in the quantized NCYM system. If this is shown, it will give a strong evidence that supports the entire matrix model as constructive definition of superstring. The structures of deformation of the open string algebra due to closed string background and the background independence [8] might also be further demonstrated.

However, since we do not know noncommutative quantum field theory itself so well, we are inclined to begin with examination of simpler systems,

and capture the generic aspects possessed by noncommutative quantum field theory.

In the succeeding sections, we would like to see a few remarkable features of the perturbative noncommutative field theory.

3 Perturbative analysis of noncommutative field theory

In order to figure out the basic facets of the perturbative framework [9] of noncommutative quantum field theory, we pay our attention to the noncommutative extension of a real scalar ϕ^4 theory in Euclidean four-dimensional space (see Ref. [2, 10] on its detail)

$$S_{\phi^4} = \int d^4x \left[\frac{1}{2} \partial_\mu \phi * \partial_\mu \phi + \frac{1}{2} m^2 \phi * \phi + \frac{\lambda}{4} \phi * \phi * \phi * \phi \right]. \quad (6)$$

The procedure for perturbation theory is the same as that in the ordinary field theory. The first task is to derive Feynman rule from the action (6). Then we apply Feynman rule to write down the diagrams relevant to the process and to the order of the coupling constant, and evaluate the associated contributions.

To derive Feynman rule, it is convenient to work in momentum space:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot x} \tilde{\phi}(p). \quad (7)$$

Then the star product works on the basis elements $e^{ip \cdot x}$ in such a way that

$$\begin{aligned} e^{ip \cdot x} * e^{iq \cdot x} &= e^{\frac{1}{2i} \partial \wedge \partial'} e^{ip \cdot x} e^{iq \cdot x'} \\ &= e^{\frac{i}{2} p \wedge q} e^{i(p+q) \cdot x}, \end{aligned} \quad (8)$$

where $p \wedge q \equiv p_\mu C^{\mu\nu} q_\nu = -q \wedge p$. This extra phase factor is reminiscent of noncommutativity of the star product.

First, we consider the propagator. Due to the total momentum conservation of the system, only one momentum is linearly independent. Thus, there is no room for phase factors to enter; the propagator is the same as in the ordinary field theory

$$\langle \tilde{\phi}(p) \tilde{\phi}(q) \rangle = (2\pi)^4 \delta^4(p+q) \frac{1}{p^2 + m^2}. \quad (9)$$

However, the interaction vertex picks up nontrivial phase factor

$$\int \prod_{j=1}^4 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_4) \times \frac{\lambda}{4} \exp \left(\frac{i}{2} \sum_{i < j} p_i \wedge p_j \right) \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_4), \quad (10)$$

from the star-product. Due to this phase factor, the interaction has only cyclic symmetry, in contrast to the point vertex in the ordinary real scalar ϕ^4 field theory which is invariant under the whole permutation group. Such a loss of symmetry of the vertex is better described if the vertex and legs get some width. The width does not permit us to exchange the two neighboring external legs, for instance. Alternatively, if multiple lines, rather than one, are assigned to each leg, they also retain only cyclic symmetry. The ordinary Yang-Mills theory is such an example [12]. It gives natural description of the propagator based on the double line representation, each line carrying the color degrees of freedom. In fact, also to the noncommutative field theory, the double line representation will turn out to be suitable. This aspect is most crucial to observe the important fact that noncommutative field theory gives the same UV structure as that of the corresponding ordinary large N field theory.

To pursue the best picture, we attempt to write each momentum p_j as a combination of outgoing and incoming momenta

$$p_j = k_j - k_{j-1}. \quad (11)$$

($k_0 = k_4$), and examine the consequences. By drawing the flow of each new momentum k_j , we get the double line representation for the vertex as shown in Fig. 1. To see that this parametrization is natural, we rewrite the vertex (10) in terms of k_i

$$\int 4^4 \prod_{j=1}^4 \frac{d^4 k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \cdots + k_4) \times \frac{\lambda}{4} \left[e^{\frac{i}{2} k_4 \wedge k_1} \tilde{\phi}(k_1 - k_4) \right] \times \cdots \times \left[e^{\frac{i}{2} k_3 \wedge k_4} \tilde{\phi}(k_4 - k_3) \right]. \quad (12)$$

(4^4 is the Jacobian factor due to the change of the variables from p_i 's to k_j 's.) There, the expression of the phase factor has simplified: $\sum_{i < j} p_i \wedge p_j$

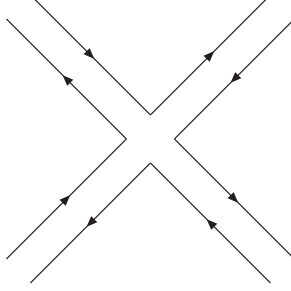


Figure 1: Double line representation for ϕ^4 -vertex.

$= \sum_{i=1}^4 k_i \wedge k_{i+1}$, and each piece has been placed in front of the field which depends on the same pair of momenta in eq. (12). Regarding the quantity in each bracket as a matrix element

$$\phi[k_1, k_2] = e^{\frac{i}{2} k_1 \wedge k_2} \tilde{\phi}(k_2 - k_1) \quad (13)$$

labeled by two momenta, it is easy to see that it constitutes a “hermitian” matrix

$$(\phi[k_1, k_2])^* = \phi[k_2, k_1], \quad (14)$$

from reality condition $\tilde{\phi}(p)^* = \tilde{\phi}(-p)$ in momentum space.

What we learn here is that, such a hermitian quantity is a building block of the interaction vertex in the noncommutative real scalar theory, and expresses Feynman rule compactly by the “double-line” representation.

Taking into account of these facts, we are inclined to recall the ordinary hermitian matrix field theory with quartic interaction

$$S_{[\Phi^4]_N} = \int d^4x \operatorname{tr} \left[\frac{1}{2} \partial_\mu \Phi \partial_\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda_H}{4N} \Phi^4 \right], \quad (15)$$

where $\Phi(x)$ is an $N \times N$ hermitian matrix-valued field. The factor $1/N$ in front of quartic interaction is prepared for the future purpose to take large N limit. In terms of the momentum space variable $\tilde{\Phi}(p)$, the interaction vertex of large N hermitian matrix field theory takes the form

$$\begin{aligned} & \int d^4x \prod_{j=1}^4 \frac{d^4k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \cdots + k_4) \\ & \times \frac{\lambda_H}{4N} \tilde{\Phi}_{i_4}^{i_1}(k_1 - k_4) \times \cdots \times \tilde{\Phi}_{i_3}^{i_4}(k_4 - k_3). \end{aligned} \quad (16)$$

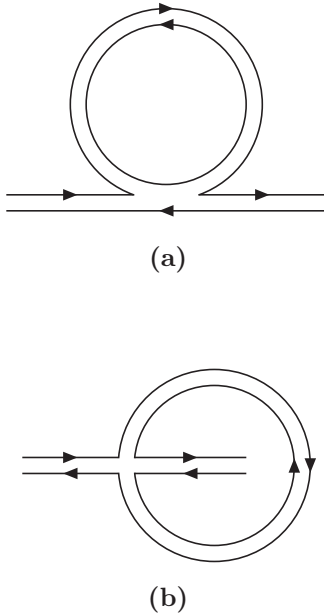


Figure 2: One-loop correction to two-point functions in ϕ^4 theory.

Comparison of eq. (12) and eq. (16) shows that Feynman diagrams drawn in noncommutative ϕ^4 theory and large N hermitian matrix field theory coincide with each other, including their combinatoric factors.

Explicit evaluation of the diagrams show that the phase factor in noncommutative field theory plays the role of the color indices carried by the matrix field in the large N field theory; the phase factor distinguishes planar and nonplanar diagrams. To illustrate this aspect in more detail, we consider one loop contribution to the two point function in both theories. There are two types of diagrams as shown in Fig. 2. As noted before, in both theories, we can draw the same diagrams. Thus, also in the side of noncommutative field theory, we can use the same terminology to distinguish these two types of the diagrams as in the ordinary field theory. That is, Fig. 2(a) is called as planar while Fig. 2(b) as nonplanar.

First we recall the situation in the side of large N field theory

$$\begin{aligned}\Pi_{\text{planar}}^{\text{large N}}(p, \lambda_H) &= \frac{\lambda_H}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}, \\ \Pi_{\text{nonplanar}}^{\text{large N}}(p, \lambda_H) &= \frac{1}{N} \frac{\lambda_H}{4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}.\end{aligned}\tag{17}$$

Both diagrams diverge quadratically, but the large N limit extracts planar one (planar limit).

We return to the side of noncommutative field theory. There, the direct computation shows that

$$\begin{aligned}\Pi_{\text{planar}}^{\text{NC}}(p, \lambda) &= \frac{\lambda}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}, \\ \Pi_{\text{nonplanar}}^{\text{NC}}(p, \lambda) &= \frac{\lambda}{4} \int \frac{d^4 q}{(2\pi)^4} e^{ip \wedge q} \frac{1}{q^2 + m^2},\end{aligned}\tag{18}$$

We see that the planar diagram in Fig. 2(a) gets no phase factors. Its contribution coincides with that of large N field theory

$$\Pi_{\text{planar}}^{\text{NC}}(p, \lambda) = \Pi_{\text{planar}}^{\text{large N}}(p, \lambda),\tag{19}$$

and diverges quadratically. However, the nonplanar diagram in Fig. 2(b) gets nonzero phase factor. We would like to see what is the effect of such a phase factor.

The Schwinger parametrization of the propagator enables us to perform the momentum integration

$$\Pi_{\text{nonplanar}}^{\text{NC}}(p, \lambda) = \frac{\lambda}{4} \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} \exp\left(-\alpha m^2 - \frac{\tilde{p}^2}{4} \frac{1}{\alpha}\right),\tag{20}$$

where $\tilde{p}^\mu = C^{\mu\nu} p_\nu$. Then the UV limit is translated to the vanishing limit of α . Nonzero noncommutative parameter ensures that the integral converges since the exponential suppression factor works when $\frac{1}{\alpha} \rightarrow \infty$. The conclusion is that nonplanar diagram is UV-finite in noncommutative field theory.

Recalling that planar diagram contributions are the same in both theories, the UV limit of noncommutative field theory is equivalent to the UV and planar limit of the corresponding large N field theory [1, 2]. This is the aspects of UV limit of noncommutative field theory. It is determined by the planar diagrams.

4 IR aspect of noncommutative field theory

Next we would like to examine IR limit of noncommutative field theory. For that purpose, we return to the nonplanar contribution (20) to the two-point

function. The simple rescaling $\alpha = \tilde{p}^2 t$ shows that this diverges in the IR side quadratically

$$\Pi_{\text{nonplanar}}^{\text{NC}}(p, \lambda) \propto \frac{1}{\tilde{p}^2} \quad \text{for } p_\mu \rightarrow 0. \quad (21)$$

To pursue its origin, we set $C^{\mu\nu}$ to zero. Then, the integral is that of the planar diagram, which diverges quadratically. It can be regularized by introducing the ultraviolet cut-off Λ [10]

$$\Pi_{\text{planar}}^{\text{NC}}(p, \lambda) \propto \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} \exp\left(\alpha m^2 - \frac{1}{4} \frac{1}{\Lambda^2} \frac{1}{\alpha}\right). \quad (22)$$

The cutoff-dependence found above reflects the quadratic divergence of the planar diagram. Comparison of the planar contribution (22) to the nonplanar contribution (20) shows that nonzero $C^{\mu\nu}$ and p_μ replace the cutoff dependence of the planar diagrams with $1/\tilde{p}^2$ in nonplanar diagrams. The IR-divergent behavior generated by the nonplanar diagrams reflects the UV-divergent behavior of the planar diagrams [10, 11].

We examine more interesting system, i.e, U(1) noncommutative Yang-Mills theory, in detail. We consider the transverse part of the renormalized vacuum polarization for the photon ¹. Its ultraviolet behavior is dominated by the planar diagrams

$$\Pi_{\mu\nu}(q)|_{\text{transverse}} \rightarrow -\frac{g^2}{16\pi^2} \frac{10}{3} \ln(q^2) (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \quad \text{for } |q| \gg \frac{1}{\sqrt{|C|}}. \quad (23)$$

The infrared limit, which is now dominated by nonplanar diagrams. can be computed [11]

$$\Pi_{\mu\nu}(q)|_{\text{transverse}} \rightarrow -\frac{g^2}{16\pi^2} \frac{10}{3} (-\ln(\tilde{q}^2)) (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \quad \text{for } |q| \ll \frac{1}{\sqrt{|C|}}. \quad (24)$$

Note that $(-\ln(\tilde{q}^2))$ is positive for $|q| \ll 1/\sqrt{|C|}$. From those results, we see that the logarithmic nature of singularities coincide with each other.

¹There arises another hard singular term proportional to the Lorentz structure, which is intrinsic to the nonvanishing noncommutativity $C^{\mu\nu}$. Its implication is discussed in Ref. [13].

Furthermore, both limiting behaviors coincide with each other, including a numerical coefficient.

There is an example which does not give such a precise correspondence between the infrared and ultraviolet sides as found in NCYM theory. It is noncommutative QED, which is a noncommutative generalization of QED [11]. Another aspect of that theory is that we cannot find its counterpart of large N field theory associated with noncommutative QED.

5 Conclusion and discussion

Here we have observed a few fundamental properties of noncommutative quantum field theory. Its UV limit is governed by planar diagrams, and usually also described by the corresponding large N field theory.

We have also seen that a new type of singularity in the infrared side is generated by the nonplanar diagrams, and it has a close relationship to the behavior at UV limit.

It is interesting to ask the practical issue whether noncommutative quantum field theory accommodates the quantum theory of gravity and string (See the recent attempts in Ref. [14]), especially whether the connection of IR and UV sides observed in NCYM theory is manifestation of some duality nature (e.g. closed-open channel duality) of string theory.

Acknowledgements

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